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SINGULAR AND FRACTIONAL INTEGRALS ALONG VARIABLE SURFACES

Dashan Fan and Shuichi Sato

Abstract. We study singular integrals associated with variable surfaces of revolution. We treat the rough kernel case where the singular integral is defined by an H^1 kernel function on the sphere S^{n-1}. We prove the L^p boundedness of the singular integral for $1 < p \le 2$ assuming that a certain lower dimensional maximal operator is bounded on L^s for all $s > 1$. We also study the (L^p, L^r) boundedness for fractional integrals associated with surfaces of revolution.

1. Introduction

For $n \ge 2$, $m \ge 1$, we let $x, y \in \mathbb{R}^n$, $x_*, y_* \in \mathbb{R}^m$ and let S^{n-1} be the unit sphere in $\mathbb{R}^n$ with the Lebesgue surface measure $d\sigma$. Let $\Omega(x)$ be a homogeneous function of degree zero on $\mathbb{R}^n$. Suppose that Ω is integrable on S^{n-1} ($\Omega \in L^1(S^{n-1})$) and

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0. \tag{1.1}$$

We study singular integrals associated with variable surfaces of revolution. Suppose $b(t)$ is a bounded function on $\mathbb{R}_+ = \{t \in \mathbb{R} : t \ge 0\}$. We consider the following singular integral operator with rough kernel:

$$Tf(x, x_*) = \text{p. v.} \int_{\mathbb{R}^n} b(|y|)\Omega(y')|y|^{-n} f(x - y, x_* - \Gamma(|y|, x_*))\, dy, \tag{1.2}$$

where f is a test function, $y' = y/|y|$ for any $y \ne 0$ and

$$\Gamma(t, x_*) = (\gamma_1(t, x_*), \gamma_2(t, x_*), \dots, \gamma_m(t, x_*))$$

is a suitable continuous mapping from $\mathbb{R}_+ \times \mathbb{R}^m$ to $\mathbb{R}^m$ such that the singular integral (1.2) exists.

Two lower dimensional maximal functions with respect to the function Γ are defined by

$$\mu_\Gamma g(x_*) = \sup_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |g(x_* - \Gamma(t, x_*))|\, dt,$$

$$M_\Gamma h(x_1, x_*) = \sup_{k \in \mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |h(x_1 - t, x_* - \Gamma(t, x_*))|\, dt,$$

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where $\mathbb{Z}$ denotes the set of all integers.

Remark 1. By the proof of Lemma 2 in [LPY], we know that if

$$\|M_\Gamma h\|_{L^p(\mathbb{R}^{1+m})} \leq C\|h\|_{L^p(\mathbb{R}^{1+m})}$$

then

$$\|\mu_\Gamma g\|_{L^p(\mathbb{R}^m)} \leq C\|g\|_{L^p(\mathbb{R}^m)}.$$

We now recall some known results in the case $m = 1$. Let $\Omega \in L^q(S^{n-1})$ with some $q > 1$. If $\Gamma(t, x_*) = t^\alpha$, $\alpha > 0$, Chen [Ch2] proved that T is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Noting that both M_Γ and μ_Γ are bounded on L^p if $\Gamma(t, x_*) = t^\alpha$, $\alpha > 0$, in [CF] Chen and Fan extended Chen's result to the case $\Gamma(t, x_*) = \gamma(t)$ for which they merely required that μ_γ is bounded on $L^p(\mathbb{R}^1)$ (see also [KWWZ]). On the other hand, if $\Gamma(t, x_*) \equiv 0$, T becomes the well-known singular integral operator which was initially defined in Calderón-Zygmund's pioneering work [CZ] and later extensively studied by many authors. Readers can see [Ch1], [F], [DR], [FP2], et al., among numerous papers in this topic. It is known in [FP2], if $\Gamma(t, x_*) \equiv 0$, that the best size condition so far on Ω is $\Omega \in H^1(S^{n-1})$, where H^1 is the Hardy space on the sphere. It should be noted that the space $L\log^+ L$, which appeared in the original work of Calderón and Zygmund [CZ], contains L^q for $q > 1$ and is a proper subspace of H^1 on the unit sphere. Inspired by the result in [FP2], very recently, Lu, Pan and Yang obtained the following theorem (see also [FS] for an alternating proof).

Theorem A [LPY] [FS]. *Let $\Gamma(t, x_*) = \gamma(t)$ be real-valued and continuously differentiable on $(0, \infty)$ and satisfy*

$$|\gamma(t) - \gamma(0)| \leq Ct^\alpha$$

for some $\alpha > 0$ and small t, where C is a constant independent of t. Let $\Omega \in H^1(S^{n-1})$ satisfy (1.1) and $b \in L^\infty(\mathbb{R}_+)$. Then T is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$, provided that M_γ is bounded on $L^p(\mathbb{R}^2)$.

In the case $m \geq 2$, under a condition which is slightly weaker than $b \in L^\infty$, Fan and Zheng [FZ] extended Theorem A to the higher dimensional cases by considering $\Gamma(t, x_*) = (\gamma(t), \ldots, \gamma_m(t))$.

The significance of the above mentioned papers is that the kernel in the singular integral contains a very weak rough condition $\Omega \in H^1(S^{n-1})$. However, we found that all the papers with rough kernel conditions in literature consider only the case $\Gamma(t, x_*)$ independent of x_*.

Thus it is interesting to study the rough singular integrals along some variable surfaces $\Gamma(|y|, x_*)$. In this paper, being inspired by a recent paper [CSWW] by Carbery, Seeger, Wainger and Wright about the L^p boundedness on μ_Γ for $m = 2$, we study such topics. Our first main result in this paper is to establish the following theorem.

Theorem 1. *Suppose that $\Omega \in H^1(S^{n-1})$ satisfies (1.1). If M_Γ is bounded on $L^q(\mathbb{R}^{m+1})$ for all $q > 1$, then T is bounded on $L^p(\mathbb{R}^{n+m})$, $1 < p \le 2$.*

Note. If $\Gamma(t, x_*) = \Gamma(t)$, then by duality we can get the conclusion of Theorem 1 for all $1 < p < \infty$.

The proof of Theorem 1 modifies some ideas in [DR]. Since we need to use the atomic decomposition of Hardy spaces, we will carefully adopt some estimates obtained in [FP1]. In order to combine the ideas in [DR] and [FP1], we also need to obtain some new estimates. After reviewing the definition of the Hardy space and giving some known lemmas in Section 2, we will present the proof of Theorem 1 in Section 3. In Section 4, we will study fractional integrals along surfaces. In certain range of α, we study the conditions of the (L^p, L^r) boundedness for the fractional integral $\mathcal{I}_\alpha f$ along the surface of revolution $(y, \Gamma(|y|))$ defined by

$$\mathcal{I}_\alpha f(x, x_*) = \int_{\mathbb{R}^n} |y|^{-n+\alpha} \Omega(y') f(x - y, x_* - \Gamma(|y|))\, dy.$$

We also get a result for a fractional integral associated with a variable surface of revolution.

In this paper, we do not intend to pursue the study of boundedness of M_Γ. What we emphasize is to establish certain theorems, which say that, under some very mild condition, the boundedness of T on $L^p(\mathbb{R}^{n+m})$ can be obtained by the L^p boundedness of M_Γ in the lower dimensional space $\mathbb{R}^{m+1}$. Thus one can automatically obtain an L^p boundedness theorem on T as soon as a new theorem on M_Γ is obtained. We remark that if $\Gamma(t, x_*) = \Gamma(t)$, then the L^p boundedness of M_Γ and its relevant operators have been extensively studied by a number of authors. See [SW] for a survey of results through 1978, and more recently [Wn], [Ne], [NVWWn1], [NVWWn2], [NVWWn3], [C1], [C2], [C3], [CaW], [Ca et al], [DR], [ChSt], [CCVWW], [St2], et al. But for a variable surface $\Gamma(|y|, x_*)$, the L^p boundedness of M_Γ is a very new topic, see [CSWW].

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Hardy space on the sphere and some lemmas

We first recall briefly the definition of the Hardy space H^1 on the sphere. Remember that the Poisson kernel on S^{n-1} is defined by, for $0 \le r < 1$ and x', $y' \in S^{n-1}$,

$$P_{ry'}(x') = \frac{1}{\omega_{n-1}} \frac{1-r^2}{|ry'-x'|^n},$$

where ω_{n-1} is the surface area of S^{n-1}. For any $f \in L^1(S^{n-1})$, we define the radial maximal function $P^+f(x')$ by

$$P^+f(x') = \sup_{0\le r<1} \left| \int_{S^{n-1}} f(y') P_{rx'}(y')\, d\sigma(y') \right|.$$

The Hardy space $H^1(S^{n-1})$ is the linear space of all $f \in L^1(S^{n-1})$ with the finite norm $\|f\|_{H^1(S^{n-1})} = \|P^+f\|_{L^1(S^{n-1})} < \infty$. The space $H^1(S^{n-1})$ has an atomic decomposition (see [Co] or [CTW]), which will be reviewed below.

A q-atom is an L^q $(1 < q \le \infty)$ function $a(x')$ that satisfies

$$\operatorname{supp}(a) \subset \{y' \in S^{n-1} : |y' - x_0'| < \rho\} \quad \text{for some } x_0' \in S^{n-1} \text{ and } \rho \in (0,1], \tag{2.1}$$

$$\int_{S^{n-1}} a(y')\, d\sigma(y') = 0, \tag{2.2}$$

$$\|a\|_q \le \rho^{(n-1)(1/q-1)}. \tag{2.3}$$

From [Co] or [CTW], we find that any $\Omega \in H^1(S^{n-1})$ with the mean zero property (1.1) has an atomic decomposition $\Omega = \sum \lambda_j a_j$, where the a_j's are q-atoms and $\sum |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$.

Remark 2. An L^q function Ω on S^{n-1} satisfying (2.1) and (2.3) is called an L^q block.

For a non-zero $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, we let $\xi/|\xi| = \xi' = (\xi_1', \dots, \xi_n')$ as above and use $\bar{\xi}$ to denote $(\xi_2, \dots, \xi_n)$ in the rest of this paper.

Suppose $n \ge 2$ and $a(\cdot)$ is an ∞-atom on S^{n-1} satisfying (2.1)–(2.3) with $x_0' = \xi' \in S^{n-1}$ in (2.1). Let $n > 2$ and

$$F_a(s) = (1-s^2)^{(n-3)/2} \chi_{(-1,1)}(s) \int_{S^{n-2}} a(s, (1-s^2)^{1/2}\tilde{y})\, d\sigma(\tilde{y}),$$

where $\tilde{y} \in S^{n-2}$. Then, we have the following estimates for F_a.

Lemma 2.1. *Up to a constant factor independent of $a(\cdot)$, $F_a(s)$ is an ∞-atom on $\mathbb{R}^1$. More precisely, there is a constant C which is independent of $a(\cdot)$ such that*

$$\operatorname{supp}(F_a) \subseteq (\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')), \tag{2.4}$$

$$\|F_a\|_\infty \le C/r(\xi'), \tag{2.5}$$

$$\int_{\mathbb{R}} F_a(s)\, ds = 0, \tag{2.6}$$

where $r(\xi') = |\xi|^{-1}|A_\rho \xi|$ *and* $A_\rho \xi = (\rho^2 \xi_1, \rho\bar{\xi})$.

Suppose $n = 2$ and $a(\cdot)$ is an ∞-atom on S^1 satisfying (2.1)–(2.3) with $x_0' = \xi' \in S^1$ in (2.1). Let

$$f_a(s) = (1 - s^2)^{-1/2}\chi_{(-1,1)}(s)\left(a(s, (1 - s^2)^{1/2}) + a(s, -(1 - s^2)^{1/2})\right).$$

By a proof similar to that of Lemma 2.1, we have

Lemma 2.2. *Up to a constant factor independent of* $a(\cdot)$, $f_a(s)$ *is a* q*-atom on* $\mathbb{R}$*, where* q *is any fixed number in the interval* $(1, 2)$*. The support of* f_a *is in the interval* $(\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi'))$ *with* $r(\xi') = |\xi|^{-1}|A_\rho\xi|$ *(*$A_\rho\xi$ *is as in Lemma* 2.1*).*

Lemma 2.1 and Lemma 2.2 are Lemma 2.1 and Lemma 2.2 in [FP1].

3. Proof of Theorem 1

By the atomic decomposition of Ω, we have

$$\|Tf\|_p \le \sum |\lambda_j| \|B_j(f)\|_p, \tag{3.1}$$

where

$$B_j(f)(x, x_*) = \int_{\mathbb{R}^n} b(|y|)|y|^{-n} a_j(y') f(x - y, x_* - \Gamma(|y|, x_*))\, dy \tag{3.2}$$

with a_j being an ∞-atom. Therefore, to prove the L^p boundedness of T, it suffices to show

$$\|B_j(f)\|_p \le C\|f\|_p \tag{3.3}$$

where C is independent of the atom a_j.

For simplicity in our argument, we denote a_j by a and $B_j(f)$ by $B(f)$. Also without loss of generality we may assume that supp(a) is the ball $B(\mathbf{1}, \rho) \cap S^{n-1}$, where $\mathbf{1} = (1, 0, \ldots, 0)$ and $B(x, r)$ denotes the closed ball in $\mathbb{R}^n$ with center at x and radius r.

Let I_k be the interval $[2^k, 2^{k+1})$ for each integer k. Then

$$B(f)(x, x_*) = \sum_{k=-\infty}^{\infty} T_k f(x, x_*),$$

where

$$T_k f(x, x_*) = \int_{\mathbb{R}^n} b(|y|)|y|^{-n} a(y') \chi_{I_k}(|y|) f(x - y, x_* - \Gamma(|y|, x_*))\, dy.$$

Let $\mathcal{F}$ be the Fourier transform acting on the x-variable. It is easy to see that $\mathcal{F}(T_k f)(\xi, x_*)$ is equal to

$$\int_{2^k \le |y| < 2^{k+1}} b(|y|)|y|^{-n} a(y') \mathcal{F}(f)(\xi, x_* - \Gamma(|y|, x_*)) e^{-i\langle y, \xi\rangle}\, dy.$$

We have the following estimates on $\mathcal{F}(T_k f)$.

Lemma 3.1. *If μ_Γ is bounded on $L^r(\mathbb{R}^m)$ for all $1 < r \le 2$, then there exists $\beta > 0$ such that*

$$\|\mathcal{F}(T_k f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \le C|2^k A_\rho \xi| \|\mathcal{F}(f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)}, \tag{$*$}$$

$$\|\mathcal{F}(T_k f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \le C|2^k A_\rho \xi|^{-\beta} \|\mathcal{F}(f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)}; \tag{$**$}$$

where $A_\rho\xi$ is as in Lemma 2.1; C is independent of $k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$ and $\rho > 0$.

Proof. We will only prove the case $n > 2$ since we can prove the case $n = 2$ essentially in the same way (using Lemma 2.2 instead of Lemma 2.1).

For any fixed $\xi \in \mathbb{R}^n$ ($\xi \neq 0$), we choose a rotation O such that $O(\xi) = |\xi|\mathbf{1} = |\xi|(1,0,\dots,0)$. Let $y' = (s, y_2', y_3', \dots, y_n')$. Then it is easy to see that $\mathcal{F}(T_k f)(\xi, x_*)$ is equal to

$$\int_{I_k} b(t)t^{-1}\mathcal{F}f(\xi, x_* - \Gamma(t,x_*)) \int_{S^{n-1}} a(O^{-1}(y'))e^{-it|\xi|\langle \mathbf{1}, y'\rangle}\, d\sigma(y')\, dt,$$

where O^{-1} is the inverse of O. Now $a(O^{-1}(y'))$ is again an ∞-atom with support in $B(\zeta',\rho) \cap S^{n-1}$, where $\zeta' = O^2\xi' = O\mathbf{1}$, since $\operatorname{supp}(a) \subseteq B(\mathbf{1},\rho) \cap S^{n-1}$. Note that $\zeta_1' = \xi_1'$ and $r(\zeta') = r(\xi')$. Thus we have

$$\mathcal{F}(T_k f)(\xi, x_*) = \int_{I_k} b(t)t^{-1}\mathcal{F}f(\xi, x_* - \Gamma(t,x_*)) \int_{\mathbb{R}} F_{a\circ O^{-1}}(s)e^{-it|\xi|s}\, ds\, dt,$$

where $F_{a\circ O^{-1}}(s)$ is the function defined in Lemma 2.1. By Lemma 2.1, without loss of generality, we may assume that $F_{a\circ O^{-1}}$ is a q-atom on $\mathbb{R}^1$ with support in $(\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi'))$ for some $q > 1$. It is easy to see that $A(s) = 2r(\xi')F_{a\circ O^{-1}}(\xi_1' + 2r(\xi')s)$ is a q-atom with support in the interval $(-1,1)$. Changing variables, we have that $\mathcal{F}(T_k f)(\xi, x_*)$ is equal to

$$\int_{I_k} t^{-1}b(t)\mathcal{F}f(\xi, x_* - \Gamma(t,x_*)) \int_{\mathbb{R}} A(s)e^{-2itr(\xi')|\xi|s}\, ds\, e^{-it\xi_1'}\, dt. \tag{3.4}$$

We define $g_\xi(x_*) = \mathcal{F}f(\xi, x_*)$. By the cancellation condition of A, we obtain that

$$\begin{aligned}
|\mathcal{F}(T_k f)(\xi, x_*)| &\le C\int_{I_k} \left| g_\xi(x_* - \Gamma(t,x_*)) \int_{\mathbb{R}} A(s)\left\{e^{-2itr(\xi')|\xi|s} - 1\right\} ds \right| t^{-1}\, dt \\
&\le C\int_{I_k} |g_\xi(x_* - \Gamma(t,x_*))|\, r(\xi')|\xi|\, dt \\
&\le C\left|2^k A_\rho \xi\right| \mu_\Gamma(g_\xi)(x_*).
\end{aligned}$$

By the L^2 boundedness of μ_Γ, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^m} |\mathcal{F}(T_k f)(\xi, x_*)|^2 \, dx_*\right)^{1/2} &\leq C2^k \, |A_\rho \xi| \, \|\mu_\Gamma g_\xi\|_{L^2(\mathbb{R}^m)} \\ &\leq C2^k \, |A_\rho \xi| \, \|g_\xi\|_{L^2(\mathbb{R}^m)},\end{aligned}$$

which proves $(*)$ of the lemma.

Using Hölder's inequality in (3.4), we have, for some $q \in (1, 2)$,

$$\begin{aligned}|\mathcal{F}(T_k f)(\xi, x_*)| &\leq C \left(\int_{I_k} |g_\xi(x_* - \Gamma(t, x_*))|^q \, t^{-1} \, dt\right)^{1/q} 2^{-k(1-1/q)} \mathcal{J}_k \\ &\leq C \left\{\mu_\Gamma(g_\xi^q)(x_*)\right\}^{1/q} 2^{-k/q'} \mathcal{J}_k,\end{aligned}$$

where $q' = q/(q-1)$ and

$$\mathcal{J}_k = \left\{\int_{I_k} \left|\int_{\mathbb{R}} e^{-2itr(\xi')|\xi|s} A(s) \, ds\right|^{q'} dt\right\}^{1/q'}.$$

After a change of variables, it is easy to see that

$$\mathcal{J}_k \leq (2r(\xi')|\xi|)^{-1/q'} \left\{\int_{\mathbb{R}} |\hat{A}(t)|^{q'} \, dt\right\}^{1/q'},$$

where $\hat{A}$ is the Fourier transform of A about the s-variable. So by the Hausdorff-Young inequality, we have $\mathcal{J}_k \leq C(r(\xi')|\xi|)^{-1/q'}$. This shows

$$\begin{aligned}|\mathcal{F}(T_k f)(\xi, x_*)| &\leq C(r(\xi')|\xi|2^k)^{-1/q'} \left\{\mu_\Gamma(g_\xi^q)(x_*)\right\}^{1/q} \\ &= C(2^k|A_\rho \xi|)^{-1/q'} \left\{\mu_\Gamma(g_\xi^q)(x_*)\right\}^{1/q}.\end{aligned} \tag{3.5}$$

Noting that $\|\{\mu_\Gamma g_\xi^q\}^{1/q}\|_{L^2(\mathbb{R}^m)} \leq C\|g_\xi\|_{L^2(\mathbb{R}^m)}$, we prove $(**)$ in the lemma.

Remark 3. Let $0 < \alpha < n$, $\Omega \in L^q(S^{n-1})$ with $q > 1$, and

$$I_{\alpha,k} f(x, x_*) = \int_{\mathbb{R}^n} |y|^{-n+\alpha} \chi_{I_k}(|y|) \Omega(y') f(x - y, x_* - \Gamma(|y|, x_*)) \, dy.$$

Since an L^q function $\Omega(y')$ on S^{n-1} can be viewed as an L^q block with S^{n-1} as its support, by checking the proof of Lemma 3.1, we can easily obtain

Lemma 3.2. *Suppose that μ_Γ is bounded on $L^r(\mathbb{R}^m)$ for all $r \in (1,2]$, and $\Omega \in L^q(S^{n-1})$, $q > 1$. Then*

$$\|\mathcal{F}(I_{\alpha,k}f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \le C\|\Omega\|_{L^q(S^{n-1})}2^{\alpha k}\|\mathcal{F}f(\xi,\cdot)\|_{L^2(\mathbb{R}^m)}. \tag{3.6}$$

Also, if $n \ge 3$, we have

$$\|\mathcal{F}(I_{\alpha,k}f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \le C\|\Omega\|_{L^q(S^{n-1})}2^{\alpha k}|2^k\xi|^{-\beta}\|\mathcal{F}f(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \tag{3.6$'$}$$

for any $\beta \in (0, 1/(q_*)')$, where $q_* = \min(q,2)$. If $n = 2$, then for any $\beta \in (0, 1/2q')$, there is a constant $C = C_\beta$ such that

$$\|\mathcal{F}(I_{\alpha,k}f)(\xi,\cdot)\|_{L^2(\mathbb{R}^m)} \le C\|\Omega\|_{L^q(S^{n-1})}2^{\alpha k}|2^k\xi|^{-\beta}\|\mathcal{F}f(\xi,\cdot)\|_{L^2(\mathbb{R}^m)}. \tag{3.6$''$}$$

The constant C in (3.6)–(3.6$''$) is independent of ξ, k, f and Ω.

In proving Lemma 3.2, the following observation is useful.

Lemma 3.3.

(1) *Let $n \ge 3$ and $\Lambda \in L^q(S^{n-1})$, $1 \le q < \infty$. Put*

$$F(s) = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} \Lambda(s,(1-s^2)^{1/2}\tilde{y})\,d\sigma(\tilde{y}).$$

Then $\|F\|_{L^q(\mathbb{R}^1)} \le C\|\Lambda\|_{L^q(S^{n-1})}$ and $\int_{-1}^1 F(s)\,ds = 0$ if $\int_{S^{n-1}} \Lambda(y')\,d\sigma(y') = 0$.

(2) *Let $\lambda \in L^1(S^1)$ (the case $n = 2$) and*

$$f(s) = (1-s^2)^{-1/2}\chi_{(-1,1)}(s)\left(\lambda(s,(1-s^2)^{1/2}) + \lambda(s,-(1-s^2)^{1/2})\right).$$

Then $\|f\|_{L^1(\mathbb{R}^1)} \le \|\lambda\|_{L^1(S^1)}$ and $\int_{-1}^1 f(s)\,ds = 0$ if $\int_{S^1} \lambda(y')\,d\sigma(y') = 0$.

Proof. We give the proof of (1) only. The proof of (2) is similar. Let $n \ge 3$ and note that $(1-s^2)^{q(n-3)/2} \le (1-s^2)^{(n-3)/2}$ for $s \in (-1,1)$. Therefore

$$\begin{aligned}
\int_{-1}^1 |F(s)|^q\,ds &\le \int_{-1}^1 (1-s^2)^{(n-3)/2}\left(\int_{S^{n-2}} \left|\Lambda(s,(1-s^2)^{1/2}\tilde{y})\right|\,d\sigma(\tilde{y})\right)^q ds \\
&\le C\int_{-1}^1 (1-s^2)^{(n-3)/2}\left(\int_{S^{n-2}} \left|\Lambda(s,(1-s^2)^{1/2}\tilde{y})\right|^q\,d\sigma(\tilde{y})\right) ds \\
&= C\int_{S^{n-1}} |\Lambda(y')|^q\,d\sigma(y'),
\end{aligned}$$

where the second inequality follows by Hölder's inequality. The equality $\int_{-1}^{1} F(s)\,ds = \int_{S^{n-1}} \Lambda\, d\sigma$ implies the second assertion. This completes the proof of Lemma 3.3.

Proof of Lemma 3.2. By checking the proof of Lemma 3.1 and applying Lemma 3.3, we can easily obtain (3.6) and (3.6′). To prove (3.6″), we follow the idea on page 551 of [DR], assuming $\|\Omega\|_{L^q(S^{n-1})} = 1$.

$$\begin{aligned}|\mathcal{F}(I_{\alpha,k}f)(\xi,x_*)| &\le C2^{\alpha k}\int_{2^k}^{2^{k+1}} t^{-1}|\mathcal{F}(f)(\xi,x_*-\Gamma(t,x_*))I_t(\xi)|\,dt\\ &\le C2^{\alpha k}\left(2^{-k}\int_{2^k}^{2^{k+1}}|\mathcal{F}(f)(\xi,x_*-\Gamma(t,x_*))|^s\,dt\right)^{1/s}\left(\int_{2^k}^{2^{k+1}} t^{-1}|I_t(\xi)|^r\,dt\right)^{1/r}\end{aligned}$$

where s is less than and sufficiently close to 2, $r = s/(s-1)$ and

$$I_t(\xi) = \int_{S^{n-1}} \Omega(y')e^{-it\langle \xi, y'\rangle}\,d\sigma(y').$$

Noting $|I_t(\xi)|^r \le C|I_t(\xi)|^2$ and that from [DR], for any $\beta < 1/2q'$

$$(*)\qquad \left(\int_{2^k}^{2^{k+1}} t^{-1}|I_t(\xi)|^2\,dt\right)^{1/r} \le C|2^k\xi|^{-\beta}$$

if r is sufficiently close to 2. Now for any $0<\beta<1/2q'$, fix an $s<2$ such that $r = s/(s-1)$ makes the above $(*)$ true. Then using the $L^{2/s}$ boundedness of μ_Γ we easily obtain (3.6″).

Now we turn to prove Theorem 1. Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the intervals $(2^{j-1},2^{j+1})$. To be precise, we choose a radial function $\Phi \in C^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}(\Phi) \subset \{x : 1/2 < |x| \le 2\}$; $0 \le \Phi(x) \le 1$ and $\Phi(x) > c > 0$ if $3/5 \le |x| \le 5/3$. We define $\Phi_j \in C^\infty(\mathbb{R})$ by $\Phi_j(t) = \Phi(2^j x)$ $(|t| = |x|)$ and require that Φ satisfies

$$\sum_{j=-\infty}^{\infty} \Phi_j(t)^2 = 1 \quad \text{for all } t \ne 0.$$

It is easy to see

$$\operatorname{supp}(\Phi_j) \subset (2^{-j-1}, 2^{-j+1}).$$

Define the multiplier operators S_j on $\mathcal{S}(\mathbb{R}^{n+m})$ (the Schwartz space) by

$$\mathcal{F}(S_jf)(\xi,x_*) = \mathcal{F}(f)(\xi,x_*)\Phi_j(|A_\rho\xi|).$$

Following the proof of Lemma in [DR], we decompose the operator $B(f)$ by

$$(3.7)\qquad B(f) = \sum_j\left(\sum_k S_{j+k}(T_k(S_{j+k}f))\right) = \sum_j \tilde{T}_j f.$$

By the Littlewood-Paley theory, for any $q \in (1,\infty)$

$$\|\tilde{T}_j f\|_{L^q(\mathbb{R}^{n+m})} \le C \left\| \left(\sum_k |T_k(S_{j+k}f)|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^{n+m})}, \tag{3.8}$$

$$\left\| \left(\sum_k |S_{j+k}f|^2 \right)^{1/2} \right\|_{L^q(\mathbb{R}^{n+m})} \le C\|f\|_{L^q(\mathbb{R}^{n+m})}. \tag{3.8$'$}$$

Note that the operators S_j depend on the linear transform A_ρ. But by checking the proof of the classical Littlewood-Paley theory (see [St1]), one easily sees that the constant C in (3.8) and (3.8$'$) is independent of A_ρ, so that it is also independent of atoms. We now give the L^2 estimate of $\tilde{T}_j f$. By (3.8) and the Plancherel theorem,

$$\begin{aligned}
\|\tilde{T}_j f\|_2^2 &\le C \sum_k \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} |T_k(S_{j+k}f)(y,y_*)|^2 \, dy \, dy_* \\
&= C \sum_k \int_{\mathbb{R}^n} |\Phi_{j+k}(|A_\rho \xi|)|^2 \int_{\mathbb{R}^m} |\mathcal{F}(T_k f)(\xi, y_*)|^2 \, dy_* \, d\xi \\
&\le C \sum_k \int_{D_{j+k}} \int_{\mathbb{R}^m} |\mathcal{F}(T_k f)(\xi, y_*)|^2 \, dy_* \, d\xi,
\end{aligned}$$

where $D_j = \{\xi : 2^{-j-1} \le |A_\rho \xi| \le 2^{-j+1}\}$. By Remark 1, the L^q boundedness of M_Γ implies the L^q boundedness of μ_Γ. So we can invoke Lemma 3.1 in our proof. If $j \ge 0$, by $(*)$ in Lemma 3.1,

$$\begin{aligned}
\|\tilde{T}_j f\|_2^2 &\le C \sum_k \int_{\mathbb{R}^m} \left(\int_{D_{j+k}} |\mathcal{F}(f)(\xi, x_*)|^2 \left(2^k |A_\rho \xi|\right)^2 \, d\xi \right) dx_* \\
&\le C 2^{-2j} \int_{\mathbb{R}^m} \left(\sum_k \int_{D_{j+k}} |\mathcal{F}(f)(\xi, x_*)|^2 \, d\xi \right) dx_* \\
&\le C 2^{-2j} \int_{\mathbb{R}^{n+m}} |\mathcal{F}(f)(\xi, x_*)|^2 \, d\xi \, dx_*,
\end{aligned}$$

which shows

$$\|\tilde{T}_j f\|_2 \le C 2^{-j} \|f\|_2. \tag{3.9}$$

Similarly, using $(**)$ in Lemma 3.1, we have for $j < 0$

$$\|\tilde{T}_j f\|_2 \le C 2^{j\beta} \|f\|_2. \tag{3.10}$$

Next, we estimate the L^q norm of $\tilde{T}_j$. We want to prove, for all $q \in (1,2)$,

$$\|\tilde{T}_j f\|_{L^q(\mathbb{R}^{m+n})} \le C\|f\|_{L^q(\mathbb{R}^{m+n})}, \tag{3.11}$$

where C is independent of the atom a.

Since $\|B(f)\|_p \le C\sum_j \|\tilde{T}_j f\|_p$, we easily complete the proof of Theorem 1 by applying interpolation to the results (3.9)–(3.11). So it remains to prove (3.11). Now by (3.8) and (3.8′), it suffices to show that

$$\left\| \left(\sum_{k=-\infty}^{\infty} |T_k(S_{j+k}f)|^2 \right)^{1/2} \right\|_{L^q} \le c \left\| \left(\sum_{k=-\infty}^{\infty} |S_{j+k}f|^2 \right)^{1/2} \right\|_{L^q}. \tag{3.12}$$

Alternatively, it suffices to show

$$\left\| \left(\sum_{k=-\infty}^{\infty} |T_k f_k|^2 \right)^{1/2} \right\|_{L^q} \le C \left\| \left(\sum_{k=-\infty}^{\infty} |f_k|^2 \right)^{1/2} \right\|_{L^q}, \tag{3.12′}$$

for any arbitrary functions $\{f_k\}$ on $\mathbb{R}^{n+m}$. We define a linear operator $\mathcal{T}$ on $\{f_k(x,x_*)\}$ by $\mathcal{T}\{f_k(x,x_*)\} = \{(T_k f_k)(x,x_*)\}$. For any $p>1$, we consider the mix norm

$$\|\mathcal{T}\{f_k(x,x_*)\}\|_{L^p(\ell^\infty,\mathbb{R}^{n+m})} = \|\sup_k |T_k f_k|\|_{L^p(\mathbb{R}^{n+m})}.$$

Remember that $|T_k f_k|$ is bounded by

$$\begin{aligned}
&\int_{2^k \le |y| < 2^{k+1}} |b(|y|)a(y')||y|^{-n}|f_k(x-y, x_* - \Gamma(|y|,x_*))|\,dy \\
&\le C\int_{2^k}^{2^{k+1}} t^{-1} \int_{S^{n-1}} |a(y')||f_k(x-ty', x_* - \Gamma(t,x_*))|\,d\sigma(y')\,dt \\
&\le C\sup_k \int_{2^k}^{2^{k+1}} t^{-1} \int_{S^{n-1}} |a(y')||f^*(x-ty', x_* - \Gamma(t,x_*))|\,d\sigma(y')\,dt,
\end{aligned}$$

where $f^*(x,x_*) = \sup_k |f_k(x,x_*)|$. Thus

$$\sup_k |T_k f_k(x,x_*)| \le C\int_{S^{n-1}} |a(y')|M_{y',\Gamma} f^*(x,x_*)|\,d\sigma(y') \tag{3.13}$$

where

$$M_{y',\Gamma} f^*(x,x_*) = \sup_{k\in\mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |f^*(x-ty', x_* - \Gamma(t,x_*))|\,dt,$$

and hence

$$\begin{aligned}\left\|\sup_k |T_k f_k|\right\|_{L^p(\mathbb{R}^{n+m})} &\le C\left\|\int_{S^{n-1}} |a(y')| M_{y',\Gamma} f^* \, d\sigma(y')\right\|_{L^p(\mathbb{R}^{n+m})} \\ &\le C\int_{S^{n-1}} |a(y')| \, \|M_{y',\Gamma} f^*\|_{L^p(\mathbb{R}^{n+m})} \, d\sigma(y').\end{aligned} \tag{3.14}$$

Now we claim that

$$\|M_{y',\Gamma} g\|_{L^p(\mathbb{R}^{n+m})} \le C\|g\|_{L^p(\mathbb{R}^{n+m})} \tag{3.15}$$

with C independent of y'. For each fixed y', choose a rotation R such that $Ry' = \mathbf{1} = (1, 0, \ldots, 0)$. Let R^{-1} be the inverse of R. For any function g, we define the function g_R by $g_R(x, x_*) = g(Rx, x_*)$ so that $g(x - ty', x_* - \Gamma(t, x_*)) = g_{R^{-1}}(Rx - t\mathbf{1}, x_* - \Gamma(t, x_*))$. Let $x = (x_1, \bar{x})$. Then

$$\begin{aligned}&\|M_{y',\Gamma} g\|^p_{L^p(\mathbb{R}^{n+m})} \\ &= \int_{\mathbb{R}^{n+m}} \left(\sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |g_{R^{-1}}(Rx - t\mathbf{1}, x_* - \Gamma(t, x_*))| \, dt\right)^p dx\, dx_* \\ &= \int_{\mathbb{R}^{n+m}} \left(\sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |g_{R^{-1}}(x - t\mathbf{1}, x_* - \Gamma(t, x_*))| \, dt\right)^p dx\, dx_* \\ &= \int_{\mathbb{R}^{n+m}} \left(\sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |g_{R^{-1}}(x_1 - t, \bar{x}, x_* - \Gamma(t, x_*))| \, dt\right)^p dx\, dx_* \\ &= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{1+m}} \left(\sup_{k\in\mathbb{Z}} 2^{-k}\int_{2^k}^{2^{k+1}} |g_{R^{-1}}(x_1 - t, \bar{x}, x_* - \Gamma(t, x_*))| \, dt\right)^p dx_1\, dx_*\, d\bar{x}.\end{aligned}$$

Let $h(x_1, x_*) = g_{R^{-1}}(x_1, \bar{x}, x_*)$. Then, the last integral above is equal to

$$\int_{\mathbb{R}^{n-1}} \|M_\Gamma h\|^p_{L^p(\mathbb{R}^{m+1})} \, d\bar{x}.$$

By the L^p boundedness of M_Γ, we have

$$\int_{\mathbb{R}^{n-1}} \|M_\Gamma h\|^p_{L^p(\mathbb{R}^{m+1})} \, d\bar{x} \le C\int_{\mathbb{R}^{n+m}} |g_{R^{-1}}(x, x_*)|^p \, dx\, dx_* = C\|g\|^p_{L^p(\mathbb{R}^{n+m})}.$$

Obviously the above constant C is independent of y'. Thus (3.15) is proved. By (3.14) and (3.15) we have

$$\|\mathcal{T}\{f_k\}\|_{L^p(\ell^\infty,\mathbb{R}^{n+m})} \le C\|\{f_k\}\|_{L^p(\ell^\infty,\mathbb{R}^{n+m})}. \tag{3.16}$$

On the other hand, for any $r>1$, we can estimate the mix norm

$$\|\mathcal{T}\{f_k\}\|^r_{L^r(\ell^r,\mathbb{R}^{n+m})}=\sum_k\int_{\mathbb{R}^{n+m}}|T_kf_k(x,x_*)|^r\,dx\,dx_*$$

as follows. We first note that

$$\begin{aligned}|T_kf_k(x,x_*)|^r&=\left|\int_{\mathbb{R}^n}b(|y|)|y|^{-n}a(y')\chi_{I_k}(|y|)f_k(x-y,x_*-\Gamma(|y|,x_*))\,dy\right|^r\\&\le C\left\{\int_{2^k\le|y|\le 2^{k+1}}|y|^{-n}\,|a(y')f_k(x-y,x_*-\Gamma(|y|,x_*))|\,dy\right\}^r\\&\le C\left\{\int_{S^{n-1}}\int_{I_k}t^{-1}\,|f_k(x-ty',x_*-\Gamma(t,x_*))|\,dt\,|a(y')|\,d\sigma(y')\right\}^r.\end{aligned}$$

Using the method of rotation similar to the estimates for (3.14) and (3.15) we have

$$\|\mathcal{T}\{f_k\}\|_{L^r(\ell^r,\mathbb{R}^{n+m})}\le C\|\{f_k\}\|_{L^r(\ell^r,\mathbb{R}^{n+m})}\quad\text{for all } r>1. \tag{3.17}$$

Now (3.12′) follows by interpolating between (3.16) and (3.17).

We note that to prove the L^2 boundedness of Theorem 1, we only use the L^q boundedness of μ_Γ. Thus we have the following:

Corollary 1. *Suppose that $\Omega\in H^1(S^{n-1})$ satisfying* (1.1). *If μ_Γ is bounded on $L^q(\mathbb{R}^m)$ for all $q>1$, then T is bounded on $L^2(\mathbb{R}^{n+m})$.*

The examples of surfaces $\Gamma(t,x_*)$ whose maximal operators μ_Γ are bounded on $L^q(\mathbb{R}^m)$ can be found in [CSWW].

As an application of the proof of Theorem 1, we can have a result for a variable kernel. Let $\Omega(x,x_*)$ be a measurable function on $\mathbb{R}^n\times\mathbb{R}^m$ which is homogeneous of degree zero with respect to the variable x. We assume that, for each $x_*\in\mathbb{R}^m$, $\Omega(\cdot,x_*)\in L^1(S^{n-1})$ and

$$\int_{S^{n-1}}\Omega(x',x_*)\,d\sigma(x')=0. \tag{3.18}$$

Let b be a bounded function on $\mathbb{R}_+$. We consider a singular integral of the following form:

$$Tf(x,x_*)=\text{p. v.}\int_{\mathbb{R}^n}f(x-y,x_*-\Gamma(|y|,x_*))b(|y|)|y|^{-n}\Omega(y',x_*)\,dy, \tag{3.19}$$

where Γ is a suitable function such that the principal value integral exists for test functions f.

Corollary 2. *Let $1 < p \le 2$ and $p/(p-1) < r < \infty$. Suppose $\Omega(x, x_*)$ satisfies* (3.18) *and*

$$\sup_{x_* \in \mathbb{R}^m} \int_{S^{n-1}} |\Omega(x', x_*)|^r \, d\sigma(x') < \infty. \tag{3.20}$$

Then the singular integral operator T defined by (3.19) *is bounded on $L^p(\mathbb{R}^{n+m})$ provided that M_Γ is bounded on $L^q(\mathbb{R}^{m+1})$ for all $q > 1$.*

Proof. Let

$$T_k f(x, x_*) = \int_{\mathbb{R}^n} f(x - y, x_* - \Gamma(|y|, x_*))\chi_{I_k}(|y|)b(|y|)\Omega(y', x_*)|y|^{-n} \, dy.$$

Since $\Omega(\cdot, x_*)$ can be regarded as an atom on S^{n-1}, by the proofs of Lemmas 3.1 and 3.2 we have, for some $\beta > 0$,

$$\|\mathcal{F}(T_k f)(\xi, \cdot)\|_{L^2(\mathbb{R}^m)} \le C \min\left\{|2^k\xi|, |2^k\xi|^{-\beta}\right\} \|\mathcal{F}f(\xi, \cdot)\|_{L^2(\mathbb{R}^m)}.$$

Therefore by the proof of Theorem 1, to complete the proof of the corollary it suffices to show the vector valued inequality

$$\|(\sum |T_k f_k|^2)^{1/2}\|_{L^q} \le C\|(\sum |f_k|^2)^{1/2}\|_{L^q}, \tag{3.21}$$

where q is chosen to satisfy $1 < q < p$ and $q' < r$. Let

$$Nf(x, x_*) = \int_{S^{n-1}} |\Omega(y', x_*)| M_{y',\Gamma} f(x, x_*) \, d\sigma(y')$$

where $M_{y',\Gamma}$ is defined as in (3.13). Then by applying the interpolation argument in the proof of Theorem 1, to prove (3.21) it suffices to prove the $L^q(\mathbb{R}^{n+m})$ boundedness of the operator N. Now by Hölder's inequality and (3.20) we have

$$\begin{aligned} Nf(x, x_*) &\le \left(\int_{S^{n-1}} |\Omega(y', x_*)|^r \, d\sigma(y')\right)^{1/r} \left(\int_{S^{n-1}} |M_{y',\Gamma} f(x, x_*)|^{r'} \, d\sigma(y')\right)^{1/r'} \\ &\le C \left(\int_{S^{n-1}} |M_{y',\Gamma} f(x, x_*)|^{r'} \, d\sigma(y')\right)^{1/r'}, \end{aligned}$$

where C is independent of x_*. Thus by Minkowski's inequality and (3.15) we have

$$\|Nf\|_{L^q(\mathbb{R}^{n+m})} \le C \left(\int_{S^{n-1}} \|M_{y',\Gamma} f\|^{r'}_{L^q(\mathbb{R}^{n+m})} \, d\sigma(y')\right)^{1/r'} \le C\|f\|_{L^q(\mathbb{R}^{n+m})}.$$

This completes the proof.

Remark 4. For $r > 0$, let Δ_r denote the collection of measurable functions $b(t)$ on $\mathbb{R}_+$ satisfying

$$\|b\|_{\Delta_r} = \left(\sup_{R>0} R^{-1} \int_0^R |b(t)|^r \, dt\right)^{1/r} < \infty.$$

Then by checking the proof, in Theorem 1 we can replace the requirement of b being bounded by a slightly less restrictive one: $b \in \cap_{r>1}\Delta_r$. Unbounded functions with a logarithmic growth will satisfy this condition.

4. Fractional integrals

In this section, we study fractional integrals $I_\alpha f$ along certain variable surfaces defined by

$$I_\alpha f(x,x_*) = \int_{\mathbb{R}^n} |y|^{\alpha-n}\Omega(y')f(x-y,x_*-\Gamma(|y|,x_*))\,dy \tag{4.1}$$

where $0<\alpha<n$ and $\Omega\in L^1(S^{n-1})$.

Theorem 2. *Let $\Omega\in L^q(S^{n-1})$, $q>1$. Put $q_*=\min(q,2)$ as above. Suppose $n\geq 3$, $0<\alpha<1/(q_*)'$; or $n=2$, $0<\alpha<1/(2q')$. If μ_Γ is bounded on $L^r(\mathbb{R}^m)$ for all $r>1$, then we have*

$$\|I_\alpha f\|_{L^2(\mathbb{R}^{n+m})}\leq C\|\Omega\|_{L^q(S^{n-1})}\left\{\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n}|f(x,x_*)|^{2n/(n+2\alpha)}\,dx\right)^{(n+2\alpha)/n}dx_*\right\}^{1/2}.$$

Proof. Choose a smooth partition of unity $\{\Phi_j\}$ as in the proof of Theorem 1. Define the multiplier operators S_j on $\mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(S_jf)(\xi,x_*)=\mathcal{F}(f)(\xi,x_*)\Phi_j(|\xi|)$. Let $I_{\alpha,k}$ be defined as in Remark 3. Following the proof of Theorem 1, we write

$$\begin{aligned}I_\alpha f(x,x_*) &= \sum_k I_{\alpha,k}f(x,x_*) \\ &= \sum_j\left(\sum_k S_{j+k}(I_{\alpha,k}(S_{j+k}f))(x,x_*)\right)=\sum_j J_{\alpha,j}f(x,x_*).\end{aligned}\tag{4.2}$$

By Plancherel's theorem,

$$\|J_{\alpha,j}f\|^2_{L^2(\mathbb{R}^{n+m})}\leq C\sum_k\int_{E_{j+k}}\int_{\mathbb{R}^m}|\mathcal{F}(I_{\alpha,k}f)(\xi,y_*)|^2\,dy_*\,d\xi,$$

where

$$E_j=\{\xi\in\mathbb{R}^n : 2^{-j-1}\leq|\xi|<2^{-j+1}\}.$$

Let $n\geq 3$. If $j>0$, by Lemma 3.2, we have

$$\begin{aligned}\|J_{\alpha,j}f\|_2^2 &\leq C\|\Omega\|^2_{L^q(S^{n-1})}\sum_k\int_{E_{j+k}}2^{2k\alpha}\int_{\mathbb{R}^m}|\mathcal{F}f(\xi,y_*)|^2\,dy_*\,d\xi \\ &\leq C\|\Omega\|^2_{L^q(S^{n-1})}2^{2\alpha(-j+1)}\int_{\mathbb{R}^m}\int_{\mathbb{R}^n}|\mathcal{F}f(\xi,y_*)|^2|\xi|^{-2\alpha}\,d\xi\,dy_*.\end{aligned}$$

Thus by the Hardy-Littlewood-Sobolev theorem, we have

$$\|J_{\alpha,j}f\|_2^2\leq C2^{-2\alpha j}\|\Omega\|^2_{L^q(S^{n-1})}\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}|f(x,y_*)|^{2n/(n+2\alpha)}\,dx\right\}^{(n+2\alpha)/n}dy_*.\tag{4.3}$$

If $j \le 0$, by Lemma 3.2 again,

$$\begin{aligned}\|J_{\alpha,j}f\|_2^2 &\le C\|\Omega\|_{L^q(S^{n-1})}^2 \sum_k \int_{E_{j+k}} 2^{2k\alpha}|2^k\xi|^{-2\beta} \int_{\mathbb{R}^m} |\mathcal{F}f(\xi, y_*)|^2\, dy_*\, d\xi \\ &\le C2^{j(2\beta-2\alpha)}\|\Omega\|_{L^q(S^{n-1})}^2 \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |\mathcal{F}f(\xi, y_*)|^2|\xi|^{-2\alpha}\, d\xi\, dy_*.\end{aligned}$$

We now have

$$\|J_{\alpha,j}f\|_2^2 \le C2^{j\theta}\|\Omega\|_{L^q(S^{n-1})}^2 \int_{\mathbb{R}^m} \left\{\int_{\mathbb{R}^n} |f(x, y_*)|^{2n/(n+2\alpha)}\, dx\right\}^{(n+2\alpha)/n} dy_* \tag{4.4}$$

where $j \le 0$ and $\theta = 2\beta - 2\alpha$. We can assume $\theta > 0$ by taking β sufficiently close to $1/(q_*)'$. Therefore Theorem 2 follows from (4.3) and (4.4) in the case $n \ge 3$. The proof of the case $n = 2$ is similar; we skip the proof.

Next, for $\Omega \in L^1(S^{n-1})$, we study the fractional integral

$$\mathcal{I}_\alpha f(x, x_*) = \int_{\mathbb{R}^n} |y|^{-n+\alpha}\Omega(y')f(x - y, x_* - \Gamma(|y|))\, dy.$$

For $1 \le p, r < \infty$, we define the mix norm $\|f\|_{p,r}$ by

$$\|f\|_{p,r} = \left\{\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} |f(x, x_*)|^p\, dx\right)^{r/p} dx_*\right\}^{1/r}.$$

Let $q > 1$ and $q_* = \min(q, 2)$ as above. For $n \ge 3$ and $n/(n-\alpha) < r \le 2$, put

$$\alpha_0 = \frac{1}{2}\left(\frac{n}{2} + \frac{1}{(q_*)'} - \left(\left(\frac{n}{2} + \frac{1}{(q_*)'}\right)^2 - \frac{4n}{(q_*)'r'}\right)^{1/2}\right),$$

which is the smaller root of the polynomial:

$$x^2 - \left(\frac{n}{2} + \frac{1}{(q_*)'}\right)x + \frac{n}{(q_*)'r'}.$$

Note that $0 < \alpha_0 \le 2/(r'(q_*)')$. Also for $n = 2$, put

$$\beta_0 = \frac{1}{2}\left(\frac{n}{2} + \frac{1}{2q'} - \left(\left(\frac{n}{2} + \frac{1}{2q'}\right)^2 - \frac{2n}{q'r'}\right)^{1/2}\right).$$

Then we have the following:

Theorem 3. *Let $1/r = 1/p - \alpha/n$ and $\Omega \in L^q(S^{n-1})$, $q > 1$. Suppose that M_Γ is bounded on $L^s(\mathbb{R}^{1+m})$ for all $s \in (1, \infty)$. If*

(i) $n \geq 3$, $n/(n-\alpha) < r \leq 2$ *and* $0 < \alpha < \alpha_0$; *or*
(ii) $n \geq 3$, $2 \leq r < \infty$ *and* $0 < \alpha < 2/(r(q_*)')$; *or*
(iii) $n = 2$, $n/(n-\alpha) < r \leq 2$ *and* $0 < \alpha < \beta_0$; *or*
(iv) $n = 2$, $2 \leq r < \infty$ *and* $0 < \alpha < 1/(rq')$,

then we have

$$\|\mathcal{I}_\alpha f\|_{r,r} \leq C\|\Omega\|_{L^q(S^{n-1})}\|f\|_{p,r}.$$

Proof. We prove the case $n \geq 3$ only since the proof for $n = 2$ is similar. By (4.2) we know

$$\|\mathcal{I}_\alpha f\|_{r,r} \leq \sum_j \|J_{\alpha,j} f\|_{r,r}, \tag{4.5}$$

where $J_{\alpha,j} f$ is defined as in (4.2) by using $\Gamma(t)$ in place of $\Gamma(t, x_*)$. By the proof in Theorem 2, we know

$$\|J_{\alpha,j} f\|_{2,2} \leq C 2^{-j\alpha}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{p,2} \quad \text{if } j > 0 \text{ and} \tag{4.6}$$

$$\|J_{\alpha,j} f\|_{2,2} \leq C 2^{j(\beta-\alpha)}\|\Omega\|_{L^q(S^{n-1})}\|f\|_{p,2} \quad \text{if } j \leq 0, \tag{4.6$'$}$$

where $p = 2n/(n + 2\alpha)$ and $\beta \in (0, 1/(q_*)')$.

On the other hand, by the Littlewood-Paley theory, for any $r \in (1, \infty)$,

$$\|J_{\alpha,j} f\|_{r,r} \cong \left\| \left(\sum_k |I_{\alpha,k}(S_{j+k} f)|^2 \right)^{1/2} \right\|_{r,r},$$

where $I_{\alpha,k} f$ is defined as in Remark 3 by using $\Gamma(t)$ in place of $\Gamma(t, x_*)$. If we can prove

$$\left\| \left(\sum_k |I_{\alpha,k}(S_{j+k} f)|^2 \right)^{1/2} \right\|_{r,r} \leq C\|\Omega\|_{L^q(S^{n-1})} \left\| \left(\sum_k \left|2^{k\alpha} S_{j+k} f\right|^2 \right)^{1/2} \right\|_{r,r}, \tag{4.7}$$

then we have

$$\|J_{\alpha,j} f\|_{r,r} \leq C 2^{-j\alpha}\|\Omega\|_{L^q(S^{n-1})} \left\| \left(\sum_k \left|2^{k\alpha} S_k f\right|^2 \right)^{1/2} \right\|_{r,r}. \tag{4.8}$$

By [FJW], we know that for each fixed x_*

$$\left\| \left(\sum_k \left| 2^{k\alpha} S_k f(\cdot, x_*) \right|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^n)} \le C \| f(\cdot, x_*) \|_{\dot{F}_r^{-\alpha,2}(\mathbb{R}^n)}, \tag{4.9}$$

where $\dot{F}_r^{-\alpha,2}$ is the Triebel-Lizorkin space and C is independent of x_*. Let R_α be the Riesz potential defined by $(R_\alpha f)\hat{}(\xi) = |\xi|^{-\alpha} \hat{f}(\xi)$. By [FJW] we know that R_α has the "lift" property, which means that $\|R_\alpha f\|_{\dot{F}_r^{0,2}} \cong \|f\|_{\dot{F}_r^{-\alpha,2}}$. Note that the $\dot{F}_r^{0,2}$ norm is equivalent to the L^r norm. Therefore, by (4.9) and the well-known Hardy-Littlewood-Sobolev theorem we have

$$\left\| \left(\sum_k \left| 2^{k\alpha} S_k f(\cdot, x_*) \right|^2 \right)^{1/2} \right\|_r \le C \| R_\alpha f(\cdot, x_*) \|_r \le C \| f(\cdot, x_*) \|_p \tag{4.10}$$

with $1/r = 1/p - \alpha/n$. Now by (4.8) and (4.10) we obtain

$$\|J_{\alpha,j} f\|_{r,r} \le C 2^{-\alpha j} \|\Omega\|_{L^q(S^{n-1})} \|f\|_{p,r} \quad \text{with } 1/r = 1/p - \alpha/n. \tag{4.11}$$

Theorem 3 (the case $n \ge 3$) now is proved by interpolating (4.6), (4.6′) (with β sufficiently close to $1/(q_*)'$) and (4.11). Thus it remains to show (4.7). Specifically, it suffices to show that for any functions $\{f_k\}$

$$\left\| \left(\sum_k |I_{\alpha,k} f_k|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^{n+m})} \le C \|\Omega\|_{L^q(S^{n-1})} \left\| \left(\sum_k \left| 2^{k\alpha} f_k \right|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^{n+m})}. \tag{4.12}$$

If $r \ge 2$, let $s = (r/2)' = r/(r-2)$. Take a non-negative $g \in L^s(\mathbb{R}^{n+m})$ such that $\|g\|_{L^s(\mathbb{R}^{n+m})} \le 1$ and

$$I := \left\| \left(\sum_k |I_{\alpha,k} f_k|^2 \right)^{1/2} \right\|_{L^r(\mathbb{R}^{n+m})}^2 = \sum_k \int |I_{\alpha,k} f_k|^2 \, g \, dx \, dx_*.$$

Then since

$$|I_{\alpha,k} f_k(x, x_*)|^2 \le C 2^{2k\alpha} \|\Omega\|_{L^q(S^{n-1})} \int_{2^k \le |y| < 2^{k+1}} |y|^{-n} |\Omega(y')| |f_k(x - y, x_* - \Gamma(|y|))|^2 \, dy,$$

we have that I is dominated by, up to a factor $C\|\Omega\|_{L^q(S^{n-1})}$ $(C > 0)$,

$$\begin{aligned} &\sum_k 2^{2\alpha k} \int_{\mathbb{R}^{n+m}} |f_k(x, x_*)|^2 \left(\int_{2^k \le |y| < 2^{k+1}} |\Omega(y')| |y|^{-n} g(x + y, x_* + \Gamma(|y|)) \, dy \right) dx \, dx_* \\ &\qquad \le \int_{\mathbb{R}^{n+m}} \sum_k 2^{2k\alpha} |f_k(x, x_*)|^2 N_\Omega g(x, x_*) \, dx \, dx_*, \end{aligned}$$

where

$$N_\Omega g(x,x_*) = \sup_k 2^{-nk} \int_{2^k \le |y| < 2^{k+1}} |\Omega(y')| g(x+y, x_* + \Gamma(|y|))\, dy.$$

By Hölder's inequality we obtain that

$$I \le C\|\Omega\|_{L^q(S^{n-1})} \left\| \left(\sum_k |2^{k\alpha} f_k|^2 \right)^{1/2} \right\|^2_{L^r(\mathbb{R}^{n+m})} \|N_\Omega g\|_{L^s(\mathbb{R}^{n+m})}. \tag{4.13}$$

Now by the $L^s(\mathbb{R}^{1+m})$ boundedness of M_Γ and the rotation method (see the proof for (3.15)), we have

$$\|N_\Omega g\|_{L^s(\mathbb{R}^{n+m})} \le C\|\Omega\|_{L^q(S^{n-1})} \|g\|_{L^s(\mathbb{R}^{n+m})} \le C\|\Omega\|_{L^q(S^{n-1})}. \tag{4.14}$$

Thus (4.12) follows from (4.13) and (4.14).

If $1 < r < 2$, (4.12) can be proved by duality (see also [FS]).

Remark 5. For $\Omega \in L^1(S^{n-1})$ and $\delta > 0$, define

$$\mathcal{I}^\delta_\alpha f(x,x_*) = \int_{\mathbb{R}^n} |y|^{-n+\alpha} \Omega(y') f(x-y, x_* - \Gamma(\delta|y|))\, dy.$$

Let $1 \le r, p < \infty$. If there exists a constant $C > 0$ independent of $\delta > 0$ such that

$$\|\mathcal{I}^\delta_\alpha f\|_{r,r} \le C\|f\|_{p,r} \tag{4.15}$$

for all test functions f, then it follows that $1/r = 1/p - \alpha/n$ provided that $\|\Omega\|_{L^1(S^{n-1})} \ne 0$. This can be seen by a dilation argument as follows (see p. 118 in [St1]). Let $f_\delta(x,x_*) = f(\delta x, x_*)$. Then we see that

$$\delta^{-\alpha - n/r} \|\mathcal{I}_\alpha f\|_{r,r} = \|\mathcal{I}^\delta_\alpha f_\delta\|_{r,r} \le C\|f_\delta\|_{p,r} = C\delta^{-n/p}\|f\|_{p,r}$$

for all $\delta > 0$. If $1/r \ne 1/p - \alpha/n$, this implies that $\|\mathcal{I}_\alpha f\|_{r,r} = 0$ for all test functions f, and hence $\|\Omega\|_{L^1(S^{n-1})} = 0$, which contradicts our assumption. Define

$$M^\delta_\Gamma h(x_1, x_*) = \sup_{k\in\mathbb{Z}} 2^{-k} \int_{2^k}^{2^{k+1}} |h(x_1 - t, x_* - \Gamma(\delta t, x_*))|\, dt.$$

It is easy to see that the operator norms of M_Γ and M^δ_Γ are uniformly comparable for all $\delta > 0$. Thus, under the hypotheses of Theorem 3, we have (4.15).

By checking the proof of Theorem 3, it is easy to see that the operator norm of $\mathcal{I}_\alpha$ depends on the function Γ only through the operator norm of M_Γ. So we have the following:

Corollary 3. *For $\Omega \in L^q(S^{n-1})$, $q > 1$, let*

$$T_{\Omega,\mathcal{P}} f(x, x_*) = \int_{\mathbb{R}^n} |y|^{-n+\alpha} \Omega(y') f(x - y, x_* - \mathcal{P}(|y|))\, dy$$

where $\mathcal{P}(t)$ is a polynomial mapping from $\mathbb{R}_+$ to $\mathbb{R}^m$. If q, p, r and α satisfy the conditions in Theorem 3, then

$$\|T_{\Omega,\mathcal{P}} f\|_{r,r} \leq C \|\Omega\|_{L^q(S^{n-1})} \|f\|_{p,r},$$

with C being independent of the coefficients of the polynomial $\mathcal{P}$.

For the boundedness of M_Γ in the case where $\Gamma(t) = \mathcal{P}(t)$, see [St2, pp. 476-478].

Remark 6. There are many papers studying the (L^r, L^p) boundedness on the fractional integrals

$$\int_{\mathbb{R}^n} |y|^{\alpha-n} \Omega(y') f(x - y)\, dy,$$

for instance, see [DL] and its references. There are also several interesting papers for studying fractional integrals along curves. We refer to the recent paper [SeW] by Seeger and Wainger, as well as references therein. Seeger and Wainger treat the general nondegenerate hypersurface case and obtain sharp bounds for the associated fractional integrals. Our case is different from their case in two aspects. First, their integral contains a cut-off function so that the singularity at the infinity disappears. Secondly, there is no rough kernel in their integral.

We also want to point out that, in this paper, we are only able to obtain estimates under a narrow range of α. Clearly it will be interesting to know if one can extend the results to all $\alpha \in (0, n)$.

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